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JAMES' QUASI-REFLEXIVE SPACE IS NOT ISOMORPHIC TO ANY SUBSPACE OF ITS DUAL

BY

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ABSTRACT

We prove that each non-reflexive subspace of J^* contains a subspace isomorphic to J^* and complemented in J^*. Consequences are that J is not isomorphic to any subspace of J^*, and that every reflexive subspace of J is contained in a complemented reflexive subspace of J.

1. In section 2 of this paper we present the solution to a conjecture of James [7]. We prove that there is no linear isomorphism of James' quasi-reflexive Banach space into its dual. This is accomplished by proving that each non-reflexive subspace of J^* contains a subspace isomorphic to J^* and complemented in J^*, and then using James' result [7] that J^* is not isomorphic to any subspace of J. This result implies the formally stronger statement, conjectured in Casazza [2], that J and J^* are incomparable to the extent that if $X \subset J$ and $Y \subset J^*$ are non-reflexive, then X and Y are not isomorphic. Since J and J^* are quasi-reflexive of order one, their non-reflexive subspaces are also quasi-reflexive of order one [4]. In [1] we proved that every non-reflexive subspace of J contains the isomorphic image of J.

In section 3 we apply the main result of section 2 to the study of reflexive subspaces of J and J^*. We show that any reflexive subspace of J (or J^*) is contained in a complemented reflexive subspace of J (J^*).

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James' space J was introduced in [5], [6], and may be defined as the Banach space of all sequences of real numbers (a_i) such that

$$\lim_{i \to \infty} a_i = 0, \qquad \text{and}$$

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$$\|(a_i)\| = \sup_{p_1<\cdots<p_n} \frac{1}{\sqrt{2}}\left(\sum_{i=1}^{n-1} |a_{p_i} - a_{p_{i+1}}|^2 + |a_{p_n} - a_{p_1}|^2\right)^{1/2} < \infty. \tag{1}$$

With this norm, J is isometric to its second conjugate space, which is the Banach space of all sequences of reals for which the squared-variation norm (1) is finite. Since finiteness of the norm implies $\lim a_i$ exists, any $x \in J^{**}$ may be written as $x = x_0 + a\mathbf{1}$, where $x_0 \in J$, $a \in \mathbf{R}$, and $\mathbf{1}$ denotes the sequence $(1,1,1,\cdots) \in J^{**}$.

Our notation is standard in Banach space theory, as may be found in [8]. If (z_n) is a sequence in a Banach space Z, we denote the closed linear span of (z_n) by $[(z_n)]$. A sequence (z_n) is termed *semi-normalized* if there is a constant $M > 0$ such that $M^{-1} \leqq \|z_n\| \leqq M$ for all n. Schauder bases (y_n) and (z_n) are said to be *equivalent* if there is a constant M such that for all scalar sequences (a_n),

$$M^{-1}\|\Sigma\, a_n y_n\| \leqq \|\Sigma\, a_n z_n\| \leqq M\|\Sigma\, a_n y_n\|.$$

We reserve the notation (e_n) for the unit vector basis of J, and (e_n^*) for the sequence of biorthogonal functionals. It is known that (e_n^*) is a basis for J^* and that the sequence (x_n) defined by $x_n = \Sigma_{i=1}^n e_i$ is a boundedly complete basis for J, with $x_n^* = e_n^* - e_{n+1}^*$ [5], [8].

Although most computations will be done in J, we shall use the following proposition concerning the norm in J^*.

PROPOSITION 1. *Let* $x^* = \Sigma_{i=1}^\infty a_i e_i^* \in J^*$. *Then*

(a) *If* $a_i \geqq 0$ *for all* i, *then* $\|x^*\| = \Sigma |a_i|$,

(b) $\|x^*\| \geqq 1/\sqrt{2}[\Sigma_{i=1}^\infty |a_i|^2]^{1/2}$.

PROOF. Statement (a) appears in [8]. We certainly have $\|x^*\| \leqq \Sigma |a_i|$, and since $\|\mathbf{1}\|_{J^{**}} = 1$, $\Sigma |a_i| = \langle \mathbf{1}, x^* \rangle \leqq \|x^*\|$.

To prove (b), notice that (1) and the inequality $(x+y)^2 \leqq 2(x^2+y^2)$ imply that for all n,

$$\left\|\sum_{i=1}^n a_i e_i\right\| \leqq \sqrt{2}\left[\sum_{i=1}^n a_i^2\right]^{1/2}.$$

Thus for all n

$$\sqrt{2}\left[\sum_{i=1}^n a_i^2\right]^{1/2} \|x^*\| \geqq \left|\left\langle x^*, \sum_{i=1}^n a_i e_i \right\rangle\right| = \sum_{i=1}^n a_i^2,$$

so that

$$\|x^*\| \geqq \frac{1}{\sqrt{2}}\left[\sum_{i=1}^n a_i^2\right]^{1/2}.$$

2. In this section we prove

THEOREM 2. *If X is a non-reflexive subspace of J^*, then X contains a subspace isomorphic to J^* and complemented in J^*.*

The main step in the proof of Theorem 2 is to show that if (z_n) is a sequence in J^* converging to zero in the weak* topology but not in the weak topology, then (z_n) has a subsequence equivalent to the unit vector basis of J^*. To this end we present several propositions concerning block basic sequences of (e_n^*).

PROPOSITION 3. *Let (z_n) be a block basic sequence in J^* with $z_n = \sum_{i=p_n+1}^{p_{n+1}} a_i e_i^*$, and suppose $\sum_{p_n+1}^{p_{n+1}} a_i = K > 0$. Then for any scalar sequence (b_n),*

(a) $$\|\Sigma b_n z_n\| \geqq K \|\Sigma b_n e_n^*\|.$$

(b) *If $a_i \geqq 0$ for all i, then*

$$\|\Sigma b_n z_n\| \leqq \sqrt{2} K \|\Sigma b_n e_n^*\|.$$

PROOF. Let $x = \Sigma c_i e_i \in J$, and observe that $\|x\| = \|\Sigma_k c_k \Sigma_{p_k+1}^{p_{k+1}} e_i\|$. Then

$$\left\langle \sum_k b_k z_k, \sum_k c_k \sum_{p_k+1}^{p_{k+1}} e_i \right\rangle = \sum_k b_k c_k \sum_{p_k+1}^{p_{k+1}} a_i$$

$$= K \sum_k b_k c_k$$

$$= K \langle \Sigma b_k e_k^*, \Sigma c_k e_k \rangle,$$

so that (a) follows by taking the supremum over $x \in J$, $\|x\| = 1$.

Now, assuming $a_i \geqq 0$ for all i, define $\bar{c}_n = (1/K) \Sigma_{p_n+1}^{p_{n+1}} a_i c_i$ for each n. It follows from (1) that

$$\left\| \Sigma \bar{c}_n \left(\sum_{p_n+1}^{p_{n+1}} e_i \right) \right\| \leqq \sqrt{2} \left\| \Sigma \left(\sum_{p_n+1}^{p_{n+1}} c_i e_i \right) \right\|,$$

and hence

$$|\langle \Sigma b_n z_n, \Sigma c_i e_i \rangle| = \left| \sum_k b_k \sum_{p_k+1}^{p_{k+1}} a_n c_n \right|$$

$$= K |\Sigma b_n \bar{c}_n|$$

$$= K |\langle \Sigma b_n e_n^*, \Sigma \bar{c}_k e_k \rangle|$$

$$\leqq K \|\Sigma b_n e_n^*\| \|\Sigma \bar{c}_k e_k\|$$

$$\leqq \sqrt{2} K \|\Sigma b_n e_n^*\| \|\Sigma c_i e_i\|.$$

Now (b) follows by taking the supremum over $x \in J$, $\|x\| = 1$.

Recall that a basis is said to be *spreading* if it is equivalent to each of its subsequences. An immediate corollary to Proposition 3 is

COROLLARY 4. *The unit vector basis* (e_n^*) *for* J^* *is spreading.*

We now consider block basic sequences equivalent to the unit vectors in l_2.

PROPOSITION 5. *Let* $y_j = \sum_{p_j+1}^{p_{j+1}} a_i e_i^*$ *be a semi-normalized block basic sequence in* J^*, *and suppose* $\sum_{i=p_j+1}^{p_{j+1}} a_i = 0$ *for all* j. *Then* (y_i) *is equivalent to the unit vector basis of* l_2.

PROOF. Since (e_n^*) is spreading, we may assume that $a_{p_n} = 0$ for all n. Let $X = [(\sum_{p_j+1}^{p_{j+1}} e_i)] \subset J$. Then $y_j \in X^\perp$ for all j. Now, X is complemented in J [3] by the projection P, where

$$P(\Sigma c_i e_i) = \sum_n c_{p_n+1}\left(\sum_{p_n+1}^{p_{n+1}} e_i\right),$$

and has complement

$$(I-P)J = [\{e_j : j \neq p_n \ \forall n\}] \approx (\Sigma \oplus J_{k(n)})_{l_2},$$

where $J_{k(n)}$ is the span of the first $k(n)$ unit vectors in J, and is here regarded as $J_{k(n)} = [(e_{p_n+1}, \cdots, e_{p_{n+1}-1})]$. Letting Q_n denote the natural projection of J onto $J_{k(n)}$, we see that $Q_n^*(I-P^*)$ is a projection of J^* (and of $X^\perp \approx (\Sigma \oplus J_{k(n)}^*)_{l_2}$) onto $J_{k(n)}^*$. Since $a_{p_n} = 0$ for all n, $Q_n^*(I-P^*)y_n = y_n$, so that $y_n \in J_{k(n)}^*$ for all n. Thus, for any scalar sequence (b_n), $\|\Sigma b_n y_n\| = [\Sigma |b_n|^2 \|y_n\|^2]^{1/2}$, where the norms are computed in $(\Sigma \oplus J_{k(n)})_{l_2}^*$. Computations using (1) and theorem 1 of [3] show that for any $x^* \in (I-P^*)J^*$,

$$\frac{1}{2\sqrt{2}}\|x^*\|_{(\Sigma\oplus J_{k(n)})_{l_2}^*} \leqq \|x^*\|_{J^*} \leqq 2\|x^*\|_{(\Sigma\oplus J_{k(n)})_{l_2}^*}.$$

Thus, since (y_n) is assumed to be semi-normalized, (y_n) is equivalent to the unit vector basis for l_2.

PROPOSITION 6. *Let* $w_j = \sum_{p_j+1}^{p_{j+1}} a_i e_i^*$ *be a semi-normalized block basic sequence in* J^*, *and suppose* $\sum_{p_n+1}^{p_{n+1}} a_i = K > 0$ *for all* n. *Then* (w_j) *is equivalent to* (e_j^*), *and* $[(w_j)]$ *is complemented in* J^*.

PROOF. It follows from Proposition 3 that $\|\Sigma b_n e_n^*\| \leqq (1/K)\|\Sigma b_n w_n\|$ for all scalar sequences (b_n).

To establish the other inequality, we must show that the convergence of a series $\Sigma b_n e_n^*$ implies the convergence of the series $\Sigma b_n w_n$. We write

$$w_n = \sum_{p_n+1}^{p_{n+1}} a_i e_i^* = \frac{K}{p_{n+1}-p_n} \sum_{p_n+1}^{p_{n+1}} e_i^* + \sum_{p_n+1}^{p_{n+1}} \left(a_i - \frac{K}{p_{n+1}-p_n} \right) e_i^*$$

$$= z_n + y_n.$$

By Proposition 1 we see that $\|z_n\| = 1$ for all n, and that there exists a constant M such that $\|y_n\| \leqq M$ for all n. If now, $\Sigma b_n e_n^*$ converges, it follows from Proposition 3 that $\Sigma b_n z_n$ converges, and from Propositions 1 and 5 that $\Sigma b_n y_n$ converges. Hence $\Sigma b_n w_n$ is convergent. Thus (w_n) is equivalent to (e_n^*).

Using the equivalence of (w_n) and (e_n^*), it follows that the averaging projection $Q: J \to J$ defined by

$$Q(\Sigma c_n e_n) = \frac{1}{K} \sum_n \left(\sum_{i=p_n+1}^{p_{n+1}} a_i c_i \right) \left(\sum_{i=p_n+1}^{p_{n+1}} e_i \right)$$

is bounded. A simple calculation shows that $P = Q^*$ is given by

$$Q^*(\Sigma d_n e_n^*) = \sum_n \left(\frac{1}{K} \sum_{p_n+1}^{p_{n+1}} d_j \right) w_n,$$

and hence $[(w_n)]$ is complemented by P.

We are now prepared for the

Proof of Theorem 2. Let $Y \subset J^*$ be non-reflexive. Then there exists a sequence of norm one vectors $(w_n') \subset Y$ having no limit in the weak topology on J^*. Since the ball of J^* is w^* compact, we may assume, by passing to a subsequence, that (w_n') has a weak* limit $w \in J^*$.

We consider the first case when $w \in Y$, and in this case, we may assume $w = 0$. Since $w_n' \not\to 0$ weakly, there exists $x \in J^{**}$ such that $\langle x, w_n' \rangle \not\to 0$. Since $x = x_0 + \lambda 1$ for some $x_0 \in J$ and some scalar λ, it follows from the weak* convergence of w_n' to zero that $\lambda \neq 0$ and that $\langle 1, w_n' \rangle \not\to 0$. Using the weak* convergence of (w_n') to zero, the fact that $\langle 1, w_n' \rangle \not\to 0$, standard perturbation arguments, and by passing to a subsequence, there exists a block basic sequence (w_n), $w_n = \Sigma_{p_n+1}^{p_{n+1}} a_i e_i^*$ such that $\langle 1, w_n \rangle = \Sigma_{p_n+1}^{p_{n+1}} a_i = K > 0$, and such that $\Sigma \|w_n - w_n'\|$ is small enough so that $[(w_n')]$ is equivalent to $[(w_n)]$ and complemented in J^*. In this case $[(w_n')]$ is the desired subspace, since (w_n) is equivalent to (e_n^*) by Proposition 8.

We now consider the case when Y contains no sequence which is not weakly convergent yet does converge to zero in the weak* topology on J^*. Then Y contains a non-weakly-convergent sequence of norm one vectors (w_n) with weak* limit $w \notin Y$. By the preceding arguments, there exists a sequence

$(z_n) \in Y \oplus [w]$ such that $z_n \xrightarrow{w^*} 0$, $z_n \not\to 0$ weakly, (z_n) is equivalent to (e_n^*), and $[(z_n)]$ is complemented in J^*. We may write $z_n = y_n + a_n w$ with $y_n \in Y$ and $a_n \in \mathbf{R}$. Since Y contains no sequence converging weak* to zero but failing to converge weakly to zero, we may assume (a_n) has a nonzero cluster point a. By perturbing and passing to a subsequence, we may assume $z_n = y_n + ay$. Now $z_n - z_{n+1} = y_n - y_{n+1} \in Y$, and $(z_n - z_{n+1})$ is equivalent to $(e_n^* - e_{n+1}^*)$. But the sequence $(e_n^* - e_{n+1}^*)$ is biorthogonal to the boundedly complete basis (x_n) for J, so $[(e_n^* - e_{n+1}^*)]^* \approx J$. Since the predual of J is isomorphic to J^* [4], [7], it follows that $[(y_n - y_{n+1})] \approx J^*$, and that Y contains an isomorph of J^*. Moreover $[(y_n - y_{n+1})]$ is of codimension one in $[(z_n)]$ and hence is complemented in $[(z_n)]$. Since $[(z_n)]$ is complemented in J^*, it follows that $[(y_n - y_{n+1})]$ is complemented in J^*.

REMARK. There do exist non-reflexive subspaces of J^* for which the weak* convergence of a sequence to zero implies weak convergence to zero. An example is $[(e_n^* - e_{n+1}^*)]$.

THEOREM 7. *There is no linear isomorphism from J into J^*.*

PROOF. Suppose to the contrary that $T: J \to J^*$ is an isomorphism onto its range. Then by Theorem 2, TJ contains a subspace Y isomorphic to J^*. But then, denoting the isomorphism from J^* to Y by S, $T|_Y^{-1} S$ is an isomorphism from J^* into J, contradicting a result of James [7].

These results also imply the following formally stronger statement of non-comparability of J and J^*.

COROLLARY 8. *If $X \subset J$ and $Y \subset J^*$ are non-reflexive, then X and Y are not isomorphic.*

PROOF. If there exists a non-reflexive space $Y \subset J^*$ isomorphic to a subspace of J, then by the above arguments, J^* embeds in J, a contradiction.

3. In this section we use the results of section 2 and [1] to obtain information concerning reflexive subspaces of J and J^*.

THEOREM 9. *If $X \subset J$ ($X \subset J^*$) is reflexive, then there exists a reflexive space $R \subset J$ (J^*) such that R is complemented in J (J^*) and $X \subset R$.*

PROOF. Let Z denote the predual of J. Since X is reflexive, J/X is nonreflexive, and hence $(J/X)^* = X^\perp$ is non-reflexive. Now $X_\perp \subset Z$ is of codimension at most one in $X^\perp$ and is hence nonreflexive. Since Z is isomorphic to J^*, there exists, by Theorem 2, a subspace $Y \subset X_\perp$ with Y isomorphic to J^*

and complemented in Z by a projection P. Then X is contained in the complemented reflexive space $\ker P^*$.

The proof of the case $X \subset J^*$ is the same, using theorem 2.1 of [1] in place of Theorem 2.

REMARK. Similar arguments show that if X is a reflexive subspace of J (or of J^*), then the quotient space J/X (or J^*/X) contains a complemented isomorph of J (J^*).

COROLLARY 10. *If $X \subset J$ is reflexive, then X is isomorphic to a subspace of* $(\Sigma \oplus J_n)_{l_2}$.

PROOF. By Theorem 9, X is contained in a complemented reflexive subspace R. By a result of Casazza [2], complemented reflexive subspaces of J embed isomorphically in $(\Sigma \oplus J_n)_{l_2}$.

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